D. Degree and Cellular Homology

for a map $f: S^n \to S^n$

we get $f_*: \underset{\mathbb{Z}}{\underset{\wr\|}{\tilde{H}_n(S^n)}} \to \underset{\mathbb{Z}}{\underset{\wr\|}{\tilde{H}_n(S^n)}}$ $\quad$ ($n \neq 0$ could just use $H_n(S^n)$)

define the degree of f to be $\deg(f) = f_*(1) \in \mathbb{Z}$

note: 1) $\deg(id_{S^n}) = 1$

2) $\deg(f)$ only depends on f upto homotopy

3) if f is not surjective, then $\deg f = 0$

since if f misses a point $x \in S'$

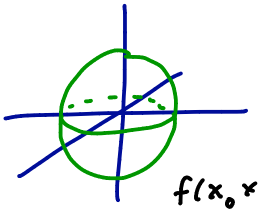
then
$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[dr, "\tilde{f} = f"'] & S^n \\ & S^n - \{x\} \arrow[u, "i"'] \end{tikzcd}$$

but $\tilde{f}_*(1) = 0 \in H_n(S^n - \{x\}) = 0$

so $f_*(1) = i_*(\tilde{f}_*(1)) = i_*(0) = 0.$

4) $\deg(f \circ g) = \deg(f)\deg(g)$

5) if f is reflection then $\deg f = -1$



$f(x_0, x_1 \ldots x_n)$
$= (-x_0, x_1, \ldots x_n)$

indeed: $n=0$ $\quad S^0 = \{-1, 1\}$ and

$f(\pm 1) = \mp 1$

$H_0(S^0) \cong H_0(\{-1\}) \oplus H_0(\{1\})$

$f_*(a,b) = (b,a)$

recall to compute reduced homology we consider

$$C_1(S^0) \xrightarrow{\partial = 0} C_0(S^0) \xrightarrow{\varepsilon} \mathbb{Z}$$
$$\textstyle\sum m_i x_i \mapsto \sum m_i$$

so $\tilde{H}_0(S^0) \cong \ker(\underset{(a,b)}{\mathbb{Z} \oplus \mathbb{Z}} \to \underset{\mapsto\ a+b}{\mathbb{Z}}) \cong \mathbb{Z}$ gen by $(1,-1)$

now $f_*(1,-1) = (-1,1) = -(1,-1)$

so $\deg f = -1$

now suppose result for S^k with $k<n$

let $D^n_{\pm} = \{(x_0, \dots x_n) \in S^n \mid \pm x_n \geq 0\}$

note f preserves $D^n_{\pm}$

$$\begin{tikzcd}
\widetilde{H}_n(S^n) \arrow[r, "\cong"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circ"] & H_n(S^n, D^n_+) \arrow[d, "f_*"] \arrow[dr, phantom, "\circ"] & H_n(D^n_-, \partial D^n_-) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circ"] & \widetilde{H}_{n-1}(\partial D^n_-) \arrow[d, "f_*"] \arrow[r, phantom, "= S^{n-1}"] & {} \\
\widetilde{H}_n(S^n) \arrow[r, "\cong", "\text{good pair Th}^{m}\,12"'] & H_n(S^n, D^n_+) & H_n(D^n_-, \partial D^n_-) \arrow[l, "\cong"', "\text{excision}"] \arrow[r, "\cong"] & \widetilde{H}_{n-1}(\partial D^n_-) \arrow[r, phantom, "= S^{n-1}"] & {}
\end{tikzcd}$$

so all vertical maps
are multiplication by -1

$$\underset{\overset{\shortparallel}{0}}{H_n(D^n_-)} \to H_n(D^n_-, \partial D^n_-) \xrightarrow[\underset{\cong}{\partial}]{} H_{n-1}(\partial D^n_-) \to \overset{n>1}{\underset{\overset{\shortparallel}{0}}{H_{n-1}(D^n_-)}}$$

if $n=1$

$$\underset{\overset{\shortparallel}{0}}{H_1(D^1_-)} \to H_1(D^1_-, \partial D^1_-) \to \underset{\overset{\shortparallel}{\mathbb{Z}\oplus\mathbb{Z}}}{H_0(S^0)} \xrightarrow{i} \underset{\overset{\shortparallel}{\mathbb{Z}}}{H_0(D^1)} \to \underset{0}{0}$$

$$H_1(D^1_-, \partial D^1_-) \cong \ker i \cong \widetilde{H}_0(S^0)$$

6) $f =$ antipodal map $= -id_{S^n}$

then $\deg(f) = (-1)^{n+1}$

this follows from exercise: $f =$ composition of $(n+1)$ reflections

Some nice applications of degree:

lemma 21:

let $f, g: X \to S^n \subseteq \mathbb{R}^{n+1}$

if $f(x) \neq -g(x)$ $\forall x \in X$, then $f \simeq g$

Proof:

$$H: X \times [0,1] \longrightarrow S^1$$
$$(x,t) \longmapsto \frac{(1-t)f(x) + t g(x)}{\|(1-t)f(x) + t g(x)\|}$$

is the homotopy (note OK since $f(x) \neq -g(x)$) ▦

Cor 22:

let $f: S^n \to S^n$

(1) if f has no fixed point, then $\deg f = (-1)^{n+1}$

(2) if there is no $x \in S^n$ s.t. $f(x) = -x$, then $\deg f = 1$

Proof: (1) apply lemma 21 to f and antipodal map and use homotopy invariance

(2) same as above but for f and id_{S^n} ▦

Cor 23:

$\boxed{\text{If } n \text{ is even, then any map } f: S^n \to S^n \text{ has a fixed point or an antipodal point } (x \text{ s.t. } f(x) = -x)}$

Proof: If not then $\deg f = 1$ and -1 ⊗ ▦

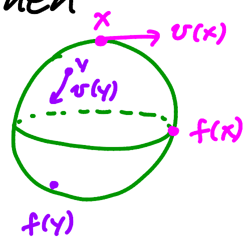
Cor 24:

$$\boxed{\begin{gathered} S^n \text{ has a nonzero vector field} \\ \Longleftrightarrow \\ n \text{ is odd} \end{gathered}}$$

Proof: If n is even then any vector field must have a zero since if v a vector field with no zero then

$$f: S^n \to S^n : x \mapsto \frac{v(x)}{\|v(x)\|}$$

has no fixed points or antipodal points ⊗



$v(x)$ lies in plane $\perp$ to x

if $n = 2k+1$, then

$$v(x_0, x_1, \dots, x_{2k}, x_{2k+1}) = (x_1, -x_0 \dots, x_{2k+1}, -x_{2k})$$

an nonzero vector field.

Remark: Its actually true that maps

$$\boxed{\begin{gathered} f, g: S^n \to S^n \text{ are homotopic} \\ \Longleftrightarrow \\ \deg f = \deg g \end{gathered}}$$

How to compute degree:

Suppose $f: S^n \to S^n$ $n > 0$

and $\exists y \in S^n$ s.t. $f^{-1}(y) =$ finite set of points $x_1, \dots, x_k$

note: $\underset{\underset{0}{\shortparallel}}{H_n(S^n - \{y\})} \to H_n(S^n) \xrightarrow{1_*} H_n(S^n, S^n - \{y\}) \to \underset{\underset{0}{\shortparallel}}{H_{n-1}(S^n - \{y\})}$ $n > 1$, think about $n = 1$ case

so 1_* an isomorphism

similarly

$J_*: H_n(S^n) \to H_n(S^n, S^n - \{x_1\})$ an isomorphism too

let V be a neighborhood of y and
U_i be neighborhoods of the x_i
s.t. $f(U_i) \subset V$ and
$x_j \notin U_i \quad \forall\, i \neq j$

by excision $H_n(S^n, S^n-\{y\}) \cong H_n(S^n-(S^n-V), (S^n-\{y\})-(S^n-V))$
$= H_n(V, V-\{y\})$

Similarly for $H_n(U_i, U_i-\{x_i\})$

so we get

$$\begin{tikzcd} H_n(U_i, U_i-\{x_i\}) \arrow[r, "f_*"] \arrow[d, phantom, "\wr\|"] & H_n(V, V-\{y\}) \arrow[d, phantom, "\wr\|"] \\ \mathbb{Z} & \mathbb{Z} \\ 1 \arrow[r, mapsto] & d \end{tikzcd}$$

note: $\cong$ with $\mathbb{Z}$ comes from $\cong$ to $H_n(S^n)$ and can fix generator there so d well-def

we define the local degree of f at x_i to be

$$\deg(f, x_i) = f_*(1) \text{ above}$$

note:
- If we change V get same number
- " " U_i " " (as long as $x_j \notin U_i$!)
- If $f|_{U_i} : U_i \to f(U_i)$ a homeomorphism then replace V by $f(U_i)$ and

$$\begin{tikzcd} H_n(U_i, U_i-\{x_i\}) \arrow[r, "f_*"] \arrow[d, phantom, "\wr\|"] & H_n(V, V-\{y\}) \arrow[d, phantom, "\wr\|"] \\ \mathbb{Z} & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \pm 1 \end{tikzcd}$$

so $\deg(f, x_i) = \pm 1$

i.e. f local homeomorphism near x_i, then $\deg(f, x_i) = \pm 1$

lemma 25:

with $f: S^n \to S^n$, y and $x_1, \dots x_k$ as above

$$\deg(f) = \sum_{i=1}^{k} \deg(f, x_i)$$

Proof:

Choose all U_i disjoint

set $Z = S^n - \bigcup_{i=1}^{k} U_i$

$$H_n(S^n, S^n - f^{-1}(y)) = H_n(S^n, S^n - \{x_1 \ldots x_k\}) \overset{\text{excision}}{\cong} H_n(S^n - Z, S^n - \{x_1 \ldots x_k\} - Z)$$
$$= H_n\Big(\bigcup_{i=1}^k U_i, \bigcup_{i=1}^k (U_i - \{x_i\})\Big)$$
$$\cong \bigoplus_{i=1}^k H_n(U_i, U_i - \{x_i\})$$

now

$$1 \longmapsto \deg(f)$$

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d, "i_*"] \arrow[dd, bend right, "g"'] \arrow[dr, phantom, "\circ"] & H_n(S^n) \arrow[d, "i_*\ \cong"] \\
H_n(S^n, S^n - f^{-1}(y)) \arrow[r, "f_*"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circ"] & H_n(S^n, S^n - \{y\}) \arrow[d, "\cong"] \\
\bigoplus_{i=1}^k H_n(U_i, U_i - \{x_i\}) \arrow[r, "\oplus (f|_{U_i})_*"'] & H_n(V, V - \{y\})
\end{tikzcd}$$

note:

$$\begin{tikzcd}
H_n(S^n) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(U_i, U_i - \{x_i\}) \arrow[d, phantom, "\cong"] \\
\mathbb{Z} & \mathbb{Z} \\
1 \arrow[r, mapsto] & 1
\end{tikzcd}$$

so $g(1) = (1, 1, \ldots 1)$

and $\deg f = f_*(1) = (\oplus (f|_{U_i})_*) \circ g(1) = \oplus (f|_{U_i})_*(1) = \sum_{i=1}^k \deg(f, x_i)$ ▦

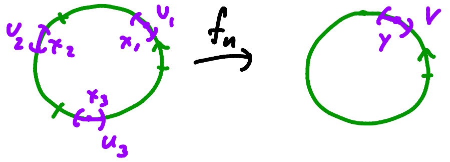
Remark: if you know differential topology then given a smooth map $f: S^n \to S^n$ we can homotop f so y is a regular value $\Rightarrow f^{-1}(y)$ finite and f local homeomorphism

$$df_{x_i}: \underset{\mathbb{R}^n}{\underset{\cong}{T_{x_i} S^n}} \to \underset{\mathbb{R}^n}{\underset{\cong}{T_y S^n}} \quad \text{isomorphism}$$

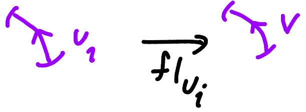
$$\deg(f, x_i) = \begin{cases} +1 & df_{x_i} \text{ orientation preserving} \\ -1 & df_{x_i} \text{ orientation reversing} \end{cases}$$

examples:

1) $f_n: S^1 \to S^1$
$z \mapsto z^n$ $\quad n > 0$



can choose so $f|_{U_i}: U_i \to V$ a homeo.



can extend $f|_{U_i}$ to a homeo $g_i: S^1 \to S^1$ that preserves or^n

such homeos are isotopic to id_{S^1}

$$\therefore \quad 1 = \deg g_1 = \deg(g_1, x_1) = \deg(f_n, x_1)$$

$$\text{so} \quad \deg f_n = n$$

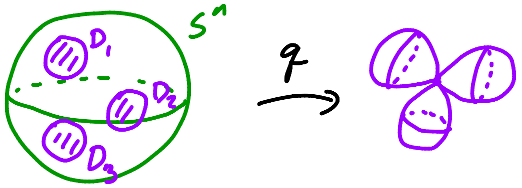
if $n<0$, then $f_n = f_{-n} \circ r$ ↙ reflection

$$\text{so} \quad \deg f_n = \deg(f_{-n}) \deg r = (-n)(-1) = n$$

2) let $D_1, \dots, D_k$ be disjoint D^n in S^n $\quad n>1$

$$C = S^n - \bigcup_{i=1}^{k} D_i$$

then $S^n/C \cong \overbrace{S^n \vee S^n \vee \dots \vee S^n}^{X}$ ↙ wedge of k, n-spheres

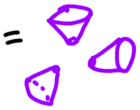


$q : S^n \to X$

let U be a nbhd of wedge pt. in X

let $V = X -$ wedge point

note: $U \cap V =$



$\cong \coprod_{i=1}^{k} (S^{n-1} \times (0,1))$

$$\begin{tikzcd} H_n(U) \oplus H_n(V) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_n(X) \arrow[r] & H_{n-1}(U \cap V) \arrow[r] \arrow[d, phantom, "\cong"] & H_{n-1}(U) \oplus H_{n-1}(V) \arrow[d, phantom, "\shortparallel"] \\ 0 & & \bigoplus_{i=1}^{k} \mathbb{Z} & 0 \end{tikzcd}$$

$$\text{so} \quad H_n(X) \cong \bigoplus_{i=1}^{k} \mathbb{Z}$$

let $f_i : X \to S^n$ collapse all but i^{th} S^n in X

Claim: $(f_i)_* : H_n(X) \longrightarrow H_n(S^n)$

$$\begin{tikzcd} H_n(X) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(S^n) \arrow[d, phantom, "\cong"] \\ \bigoplus \mathbb{Z} & \mathbb{Z} \\ (m_1, \dots m_k) \arrow[r, mapsto] & m_i \end{tikzcd}$$

indeed let $S^n \xrightarrow{j_i} X \xrightarrow{f_i} S^n$

↑ inc S^n to i^{th} sphere

$f_i \circ j_i$ is a homeomorphism

so $(f_i)_* \circ (j_i)_* (1) = \pm 1$

so $(f_i)_* (1) = \pm 1$ if -1 compose f_i with reflection now set $f : X \to S^n$ to be f_1 on i^{th} sphere

so $f_*(m_1, \dots m_k) = m_1 + \dots + m_k$

as above $q_* : H_n(S^n) \to H_n(X)$

$1 \longmapsto (1,1,\dots,1)$ **exercise:** prove this if not clear (consider ex 1) above)

set $g_k = f \circ q : S^n \to S^n$

clearly $\deg(g_k) = k$

Cellular Homology

let X be a CW complex

set $C_n^{cw}(X) =$ free abelian group generated by n-cells $e_1^n, \dots e_{\ell_n}^n$

let $f_i^n : \partial e_i^n \longrightarrow X^{(n-1)}$ the attaching map for e_i^n

given e_i^n and e_j^{n-1}, $n \geq 2$, consider

quotient onto j^{th} S^{n-1}

$$\begin{tikzcd}
S^{n-1} = \partial e_i^n \arrow[r, "f_i^n"] \arrow[rrr, bend right, "g_{ij}"'] & X^{(n-1)} \arrow[r, "\text{quotient map}"'] & X^{(n-1)}/X^{(n-2)} \cong \bigvee_{i=1}^{\ell_{n-1}} S^{n-1} \arrow[r, "p_j"] & S^{n-1}
\end{tikzcd}$$

let $d_{ij} =$ degree g_{ij}

define $\partial_n^{cw} : C_n^{cw}(X) \longrightarrow C_{n-1}^{cw}(X)$

$$e_i^n \longmapsto \sum_{j=1}^{\ell_{n-1}} d_{ij} e_j^{n-1}$$

for $n = 1$ define:

$$\partial_1^{cw} : C_1^{cw}(X) \to C_0^{cw}(X)$$

$$e_i^1 \longmapsto \partial e_i^1$$

singular boundary since $e_i^1 \hookrightarrow X$ is a sing 1-simplex

note: if $X^{(0)} = \{\text{one point}\}$, then $\partial_1^{cw} e_i^1 = 0 \quad \forall i$

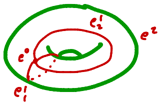
Th$^{\underline{m}}$ 26:

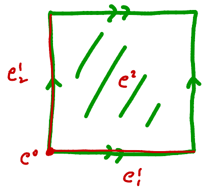
$$\boxed{\begin{array}{l}\partial_n^{cw} \circ \partial_{n+1}^{cw} = 0 \\ H_n(X) \cong \ker \partial_n^{cw} / \operatorname{im} \partial_{n+1}^{cw}\end{array}}$$

$H_n^{cw}(X) = \ker \partial_n^{cw} / \operatorname{im} \partial_n^{cw}$ is called the cellular homology of X

and th$^{\underline{m}}$ says $H_n^{cw}(X)$ is isomorphic to singular homology!

example:

1) $T^2 =$ 



$$\begin{tikzcd} 0 \arrow[r] & C_2^{cw}(T^2) \arrow[r, "\partial^{cw}"] & C_1^{cw}(T^2) \arrow[r, "\partial^{cw}"] & C_0^{cw}(T^2) \arrow[r] & 0 \end{tikzcd}$$

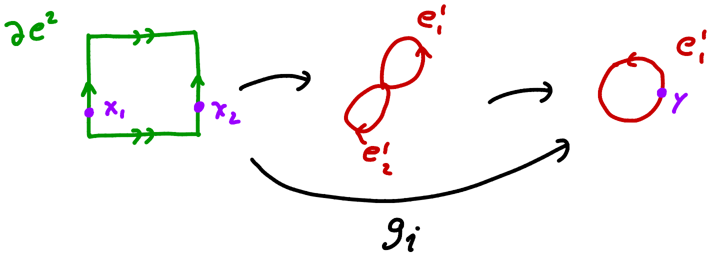
$$C_2^{cw}(T^2) = \mathbb{Z}, \quad C_1^{cw}(T^2) = \mathbb{Z} \oplus \mathbb{Z}, \quad C_0^{cw}(T^2) = \mathbb{Z}$$

for $\partial^{cw} e_i^1 = 0$ from above

for $\partial^{cw} e^2$:

$$\begin{tikzcd} \partial e^2 = S^1 \arrow[r] \arrow[rrr, bend right, "g_i"'] & X^{(1)} \arrow[r] & X^{(1)}/X^{(0)} = X^{(1)} \arrow[r] & S^1_i \end{tikzcd}$$

corresp to e_i^1



note: orientation on ∂e^2 agrees with direction at x_2 but not at x_1

so as discussed above

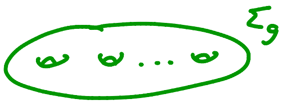
$$\deg(g_i, x_2) = 1 = -\deg(g_i, x_1)$$

so $\deg(g_i) = 0$ for $i = 0,1$

$$\therefore \partial_1^{cw} e^2 = 0 e_1^1 + 0 e_2^1 = 0$$

$$\therefore \; H_n(T^2) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{otherwise} \end{cases}$$

exercise: If Σ_g is surface of genus g



then
$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & n=0,2 \\ \bigoplus_{2g}\mathbb{Z} & n=1 \\ 0 & \text{otherwise} \end{cases}$$

Remarks: 1) $H_k(X)$ has at most $\ell_k = \#$ k-cells generators

in particular, $H_k(X)=0$ if no k-cells

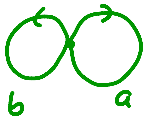
2) If X has only cells in even dimensions then $\partial^{cw}=0$

so $H_n^{cw}(X) = C_n^{cw}(X)$

example: recall $\mathbb{C}P^n = e^0 \cup e^2 \cup e^4 \cup \ldots \cup e^{2n}$

so
$$H_n^{cw}(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & n=0,2,\ldots,2n \\ 0 & \text{otherwise} \end{cases}$$

example: let $X =$



$\cup\ 2\,(2\text{-cells})$

e_1^2 attached along $a^5 b^{-3}$

e_2^2 " " $b^3(ab)^{-2}$

arguing as above we have

$$\begin{tikzcd} 0 \arrow[r] & C_2^{cw}(X) \arrow[r, "\partial_2^{cw}"] \arrow[d, phantom, "\cong" rotate=90] & C_1^{cw}(X) \arrow[r, "\partial_1^{cw}"] \arrow[d, phantom, "\cong" rotate=90] & C_0^{cw}(X) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\ & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z} & \end{tikzcd}$$

$$\partial_1^{cw} = 0$$

$$\partial_2^{cw} = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$$

note matrix invertable over $\mathbb{Z}$ so $\ker \partial_2^{cw} = 0$

$\operatorname{im} \partial_2^{cw} =$ everything

$$\therefore \; H_n^{cw}(X) \cong \begin{cases} \mathbb{Z} & n=0 \\ 0 & n\neq 0 \end{cases}$$

by Van Kampen $\pi_1(X) \cong \langle a, b \mid a^5 b^{-3}, b^3 (ab)^{-3} \rangle$

one can show this is a group of order 120

so X not contractible

note: example shows π_1 sees things H_n does not

but $\pi_1(S^n) = 0, n > 1$ so H_n sees things π_1 does not.

lemma 27:

X a CW complex

1) $H_k(X^{(n)}, X^{(n-1)}) = \begin{cases} \bigoplus_{\ell_n} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$ $\quad \ell_k = \#$ n-cells

2) $H_k(X^{(n)}) = 0$ if $k > n$

3) $\imath : X^{(n)} \to X$ induces an isomorphism

$\imath_* : H_k(X^{(n)}) \to H_k(X) \quad \forall\, k < n$

Proof: 1) $(X^{(n)}, X^{(n-1)})$ is a good pair so

$$H_k(X^{(n)}, X^{(n-1)}) \cong \tilde{H}_k(X^{(n)}/X^{(n-1)})$$

but $X^{(n)}/X^{(n-1)} \cong \bigvee_{i=1}^{\ell_k} S^n$ for $n \geq 1$

for $n = 0$ also clearly $\underline{\text{true}}$ ✓

2) $$\underset{\underset{0}{\parallel}}{H_{k+1}(X^{(n)}, X^{(n-1)})} \to H_k(X^{(n-1)}) \to H_k(X^{(n)}) \to \underset{\underset{0}{\parallel}}{H_k(X^{(n)}, X^{(n-1)})}$$

$k \neq n, n-1$

$\therefore H_k(X^{(n-1)}) \cong H_k(X^{(n)}) \quad \forall\, k \neq n, n-1$

so for $k > n$ $\quad H_k(X^{(n)}) \cong H_k(X^{(n-1)}) \cong \ldots \cong H_k(X^{(0)}) = \underline{0}$ ✓

3) if $k < n$ then

$\swarrow$ m arbitrary

$$H_k(X^{(n)}) \cong H_k(X^{(n+1)}) \cong \ldots \cong H_k(X^{(n+m)})$$

so $H_k(X^{(n)}) \cong H_k(X)$ (clear if X finite dim'l still true for any X but need **Fact:** "Homology commutes with direct limits" (colimits)

Proof of Th$^{\underline{m}}$ 26:

by lemma 25 we know $C_n^{cw}(X) \cong H_n(X^{(n)}, X^{(n-1)})$

consider the long exact sequence of the tripple $(X^{(n+1)}, X^{(n)}, X^{(n-1)})$

$$\cdots \to H_{n+1}(X^{(n+1)}, X^{(n-1)}) \to H_{n+1}(X^{(n+1)}, X^{(n)}) \xrightarrow{d_{n+1}} H_n(X^{(n)}, X^{(n-1)}) \to \cdots$$

so $d_{n+1}: C_{n+1}^{cw}(X) \to C_n^{cw}(X)$

Claim: $\partial_n^{cw} = d_n$

we prove claim below, but first prove th$^{\underline{m}}$ given claim

consider 2 long exact sequences of pairs $(X^{(n+1)}, X^{(n)})$ and $(X^{(n)}, X^{(n-1)})$

$$\begin{tikzcd}
 & H_n(X^{(n-1)}) = 0 \arrow[d] & & \text{by lemma 27} \\
H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "\partial_{n+1}"] \arrow[rdd, bend right, "j_n \circ \partial_{n+1}"'] & H_n(X^{(n)}) \arrow[r] \arrow[dd, "j_n"] & H_n(X^{(n+1)}) \arrow[r] \arrow[d, phantom, "\cong" sloped] & H_n(X^{(n+1)}, X^{(n)}) \arrow[d, phantom, "=" sloped] \\
 & & H_n(X) & 0 \\
 & H_n(X^{(n)}, X^{(n-1)}) \arrow[d, "\partial_n"] & & \\
 & H_{n-1}(X^{(n-1)}) & &
\end{tikzcd}$$

exercise: $j_n \circ \partial_{n+1} = d_{n+1}$

(diagram chase, easy to see choices maid to construct ∂_{n+1} can also be used for d_{n+1})

so $d_n \circ d_{n+1} = j_{n-1} \circ \underbrace{\partial_n \circ j_n} \circ \partial_{n+1} = 0$

$= 0$ since 2 terms in long exact sequence

$\therefore$ can consider $\ker d_n / \operatorname{im} d_{n+1}$

from above $H_n(X) \cong H_n(X^{(n)}) / \operatorname{im} \partial_{n+1}$

note: j_n is injective so

$$\operatorname{im} \partial_{n+1} \cong j_n(\operatorname{im} \partial_{n+1}) = \operatorname{im}(j_n \circ \partial_{n+1}) = \operatorname{im} d_{n+1}$$

and since $\jmath_{n-1}$ is injective too

$$H_n(X^{(n)}) \cong \operatorname{im} \jmath_n = \ker \partial_n \cong \ker(\jmath_{n-1}\circ \partial_n) = \ker d_n$$

$$\therefore\ H_n(X) \cong H_n(X^{(n)})/\operatorname{im}\partial_{n+1} \underset{\text{given by } \jmath_n}{\cong} \operatorname{im}\jmath_n/\operatorname{im} d_{n+1} = \ker d_n/\operatorname{im} d_{n+1}$$

$$\underset{\text{by claim}}{=} \ker\partial_n^{cw}/\operatorname{im}\partial_{n+1}^{cw}$$

Proof of Claim:

first note: $\imath : (e_i^n, \partial e_i^n) \to (X^{(n)}, X^{(n-1)})$ given by "inclusion"

induces

$$\begin{tikzcd}
\imath_* : H_n(e_i^n, \partial e_i^n) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_n(X^{(n)}, X^{(n-1)}) \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z} & \bigoplus_{\ell_n} \mathbb{Z}
\end{tikzcd}$$

is injective and maps $\mathbb{Z}$ to factor corresp to e_i^n

(indeed

$$\begin{tikzcd}
(e_i^n, \partial e_i^n) \arrow[r, "\imath"'] \arrow[rrr, bend right, "j"'] & (X^{(n)}, X^{(n-1)}) \arrow[r] & X^{(n)}/X^{(n-1)} \arrow[r] & e_i^n/\partial e_i^n
\end{tikzcd}$$

$$\begin{tikzcd}
H_n(e_i^n, \partial e_i^n) \arrow[r, "j_*"] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\circ"] & H_n(e_i^n/\partial e_i^n) \\
H_n(e_i^n/\partial e_i^n) \arrow[ur, "j_*"'] &
\end{tikzcd}$$

identity map)

now

$$\begin{tikzcd}
H_n(e_i^n, \partial e_i^n) \arrow[r, "\partial"] \arrow[d, "\imath_*"] \arrow[dr, phantom, "\circ"] & H_{n-1}(\partial e_i^n) \arrow[d, "(f_i^n)_*"] \\
H_n(X^{(n)}, X^{(n-1)}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] \arrow[dr, phantom, "\circ *", near end, shift left=2] & H_{n-1}(X^{(n-1)}) \arrow[d, "\jmath_{n-1}"] \\
& H_{n-1}(X^{(n-1)}, X^{(n-2)})
\end{tikzcd}$$

exercise check *

so generator in $H_n(X^{(n)}, X^{(n-1)})$ corresponding to e_i^n

maps under d_n to

$\jmath_{n-1}\circ(f_i^n)_*(1)$ in $H_{n-1}(X^{(n-1)}, X^{(n-2)})$

but $H_{n-1}(X^{(n-1)}, X^{(n-2)}) \cong \bigoplus_{\ell_{n-1}} \mathbb{Z}$

and by definition $(\jmath_{n-1}\circ f_{i*}^n)(1) = (d_{i1}, \dots, d_{i\ell_{n-1}})$

$\therefore\ d_n(\text{gen. corresp to } e_i^n) = \partial_n^{cw}(e_i^n)$ ▦

E. Homology with different coefficients

given an abelian group G
and a space X

let $C_n(X,G) = \{ \sum_{i=1}^{k} g_i \sigma_i \mid g_i \in G,\ \sigma_i$ a singular n-simplex$\}$

$$\partial_n\left(\sum_{i=1}^{k} g_i\sigma_i\right) = \sum_{i=1}^{k} g_i \partial\sigma_i = \sum_{i=1}^{k}\sum_{j=0}^{n} g_i(-1)^j \sigma_i^{(j)} \swarrow\ j^{\text{th}} \text{ face of } \sigma_i$$

as before $\partial_n \circ \partial_{n+1} = 0$

so we define the homology of X with coefficients in G to be

$$H_n(X;G) = \ker\partial_n / \operatorname{im}\partial_{n+1} \qquad (\underline{\text{note}}\text{: for } G=\mathbb{Z} \text{ get orig def}^{\underline{n}})$$

can also define $H_n(X,A;G)$ using $C_n(X,A;G) = C_n(X;G)/C_n(A;G)$

all th$^{\underline{m}}$s we proved above work for these homologies too

similarly if X a CW complex let

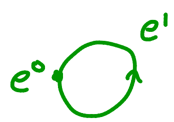
$$C_n^{cw}(X;G) = \bigoplus_{\ell_n} G \qquad \ell_n = \#\ n\text{-cells}$$

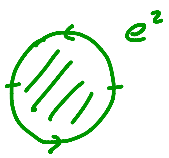
and $$\partial_n^{cw}\left(\sum_{i=1}^{\ell_n} g_i e_i^n\right) = \sum_{i=1}^{\ell_n}\sum_{j=1}^{\ell_{n-1}} g_i(\deg h_{ij}) e_j^{n-1}$$

where
$$\begin{tikzcd} \partial e_i^n \arrow[r] \arrow[rrr, bend right, "h_{ij}"'] & X^{(n-1)} \arrow[r] & X^{(n-1)}/X^{(n-2)} \arrow[r] & S^{n-1} \text{ corresp to } e_j^{n-1} \end{tikzcd}$$

again this gives $H_n(X;G)$

example: $\mathbb{R}P^2$





use $\mathbb{Z}$ coeff:

$$\begin{tikzcd} 0 \arrow[r] & C_2(\mathbb{R}P^2) \arrow[r] \arrow[d, phantom, "\cong"] & C_1(\mathbb{R}P^2) \arrow[r] \arrow[d, phantom, "\cong"] & C_0(\mathbb{R}P^2) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\ & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} & \end{tikzcd}$$

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n=1 \\ 0 & n \neq 0,1 \end{cases}$$

use $\mathbb{Z}/2$ coeff:

$$\begin{tikzcd} 0 \arrow[r] & C_2(\mathbb{R}P^2;\mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\cong"] & C_1(\mathbb{R}P^2;\mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\cong"] & C_0(\mathbb{R}P^2;\mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\ & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 & \end{tikzcd}$$

$$H_n(\mathbb{R}P^2;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n=0,1,2 \\ 0 & n \neq 0,1,2 \end{cases}$$